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Labelling schemes for tetrahedron equations and dualities between them

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Abstract. Zamolodchikov's tetrahedron equations, which were derived by considering the scattering of straight strings, can be written in three different labelling schemes: one can use as labels the states of the vacua between the strings, the states of the string segments, or the states of the particles at the intersections of the strings. We give a detailed derivation of the three corresponding tetrahedron equations and also show how the Frenkel–Moore equations fits in as a *non-local* string labelling. We then discuss how an analogue of the Wu–Kadanoff duality can be defined between each pair of the above three labelling schemes. It turns out that there are two cases, for which one can simultaneously construct a duality between *all* three pairs of labellings.

1. Introduction

Now that a quite good understanding of (1+1)-dimensional integrable systems (both classical and quantum, continuum and discrete) has been obtained, attention has turned to higher dimensions where serious difficulties have been encountered. The various approaches that were successful in 1 + 1 dimensions have different natural extensions to 2 + 1 dimensions and it is not clear which method is best. It is therefore important to push each one and hopefully they will illuminate different aspects of 2 + 1 integrable systems.

In this paper we consider the extension of the Yang–Baxter equations to 2+1 dimensions. The fundamental work in this problem was done at the beginning of the 1980's, first by Zamolodchikov [1], who derived the relevant tetrahedron equations by studying the scattering of straight strings. In this formulation it was natural to use the quantum numbers of the string segments (faces in the lattice formulation) as labels. Subsequently Bazhanov and Stroganov [2] wrote down the equations corresponding to cell and edge labelling. A different type of edge tetrahedron equation has been proposed by Frenkel and Moore [3].

Higher-dimensional generalizations of the tetrahedron equations have also been discussed in the literature. The 4-simplex equations already appeared in the above paper of Bazhanov and Stroganov [2] and *d*-simplex equations have been discussed, for example, by Maillet and Nijhoff [4] and Carter and Saito [5]. For related geometric constructions, see [6].

In all formulations the number of equations is huge even for the simplest two-state model, and subsequent progress has been slow because it has been exceedingly difficult to find solutions (especially those with spectral parameters) to these equations. The original

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solution proposed by Zamolodchikov [1] was studied in detail by Baxter [7], and only quite recently some further solutions have been found [10, 13].

Much of the work on tetrahedron equations has been done in the framework of solvable lattice models. Each formulation has its own natural properties and from time to time it is useful to look at all of them for inspiration. With this in mind we return to the original formulation of straight string scattering, and from this point of view look in detail at the properties of the various methods of labelling the tetrahedron equations.

Our main objective is to study the analogues of Wu–Kadanoff duality in the tetrahedron situation. The duality between the vertex and face formulations of the Yang–Baxter equations imposes certain restrictions for the existence of non-zero elements of the R -matrix. These restrictions amount to the very important 8-vertex ansatz, and our hope is that also in the tetrahedron case duality will lead us to fruitful ansätze.

The organization of this paper is as follows: in the next section we start by rederiving the Yang–Baxter equations in detail, because we want to use the analogies in the tetrahedron case. We also discuss in similar detail the well known Wu–Kadanoff duality that connects the two formulations of the Yang–Baxter equations under certain circumstances. In section 4 we then derive three versions of the tetrahedron equations. They differ by the choice of labels, we can use as labels the state of the vacuum between the strings, the state of the string segments, or the state of the particles at the intersection of the strings. The Frenkel–Moore equation will also be obtained if we use non-local string labelling. In section 5 we will derive a duality between each pair of tetrahedron equations, in analogue to the Wu–Kadanoff duality. This implies certain restrictions (which are explicitly written out) on the functions. When these restrictions hold both equations reduce to a common equation that has fewer labels and summation indices. Finally we consider the possibility of one equation being dual to the two others.

Our main result is the identification of two cases, for which there can simultaneously exist a duality between each pair of labelling schemes.

2. The Yang–Baxter equations

In this section we review the particle scattering derivation of the Yang–Baxter equations. This is done in some detail, because analogous steps will be followed in the derivation of the tetrahedron equations.

2.1. The basic state and the basic scattering process

The Yang–Baxter equation can be interpreted as describing scattering, with a factorizable scattering matrix, in $1+1$ dimensions [10]. The ambient space is one-dimensional and the fundamental state is a point particle moving with a constant speed, see figure 1. The particle divides the space into two parts and the state of the vacuum can be different in them (in this case the moving particle can be modelled by a kink that interpolates between the two vacua). In the general case we must therefore use three labels (in addition to a dynamical characterization by velocity) to completely describe the basic state.

The basic scattering process is obtained if we have two particles moving with different velocities. Initially the particles approach each other, scatter, and finally recede from each other, see figure 2. During scattering the momenta do not change, but the internal states of the particles can change, as well as the vacuum between them. Thus, in principle, the scattering amplitude could at the same time depend on four vacuum labels and four particle labels, in addition to the relative momenta (or the angle between the intersecting trajectories).



Figure 1. The basic state and its labelling.

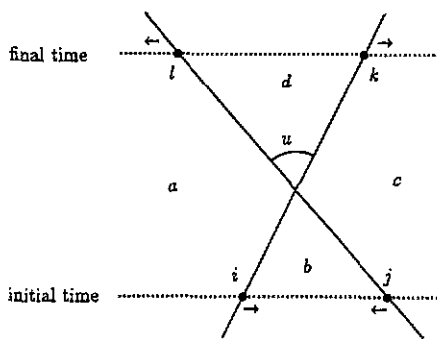


Figure 2. The basic process: Particles i and j collide resulting in particles k and l . The collision is elastic in the sense that the momenta do not change and inelastic in that the colliding particles and the vacuum between them can change.

Usually one uses only vacuum or particle labelling. Vacuum (or ‘face’) labelling is used e.g. in the ‘interaction round a face’ (IRF) formulation of lattice models [11], where the scattering amplitude of figure 2 is given by the ‘Boltzmann weight’ $w(b, c, d, a; u)$; here the first label is for the vacuum between the incoming particles and thereafter counterclockwise. In particle labelling (or vertex formulation of lattice models) one uses the R -matrix and the amplitude of the above process is R_{ij}^{kl} .

2.2. Condition from factorizability

The above describes fully what can happen with two particles. The situation becomes more interesting when we have three particles. Since the two-body scatterings are elastic (momenta do not change), the approaching particles will undergo precisely three of these pairwise scatterings before they fly out again (figure 3). The order in which they take place depends on the relative initial location of the particles, and the natural assumption is that for the three particle scattering amplitude *the order should not make any difference*. Pictorially this is stated in figure 4 [10].

The equations following from figure 4 depend on which labelling scheme we are using. If we use vacuum (face) labelling, where the scattering amplitude of figure 2 is given by $w(b, c, d, a; u)$ the condition of figure 4 gives [11]

$$\sum_k w(b, c, g, a; u)w(c, d, e, g; u + v)w(g, e, f, a; v) = \sum_k w(c, d, g, b; v)w(b, g, f, a; u + v)w(g, d, e, f; u). \tag{1}$$

There is just one summation index, the vacuum inside the triangle, which appears in all scattering amplitudes.

If we use particle (vertex) labelling and the scattering amplitude for the scattering process figure 2 is given by $R_{ij}^{kl}(u)$ then the condition of figure 4 gives

$$R_{j_1 j_2}^{k_1 k_2}(u)R_{k_1 j_3}^{l_1 k_3}(u + v)R_{k_2 k_3}^{l_2 l_3}(v) = R_{j_2 j_3}^{k_2 k_3}(v)R_{j_1 k_3}^{k_1 l_3}(u + v)R_{k_1 k_2}^{l_1 l_2}(u). \tag{2}$$

In this case there are three pairwise summation indices (k_i) so we have used Einstein’s summation convention. Because of this we can consider R as a matrix operating on the

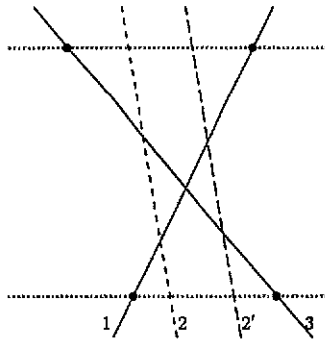


Figure 3. The two possible three particle scattering scenarios with the same momenta but different initial position of particle 2.

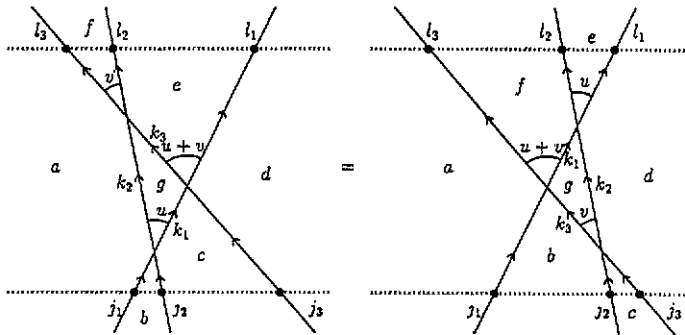


Figure 4. Pictorial representation of the factorization condition.

tensor product of two vector spaces, while the equation itself is defined on the tensor product of three vector spaces. Thus we can use the shorthand notation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{3}$$

where we have only written out the names of the vector spaces on which R operates non-trivially. The matrix interpretation has many important group-theoretical consequences, for example, it is easy to see that (2) is invariant under ‘gauge’ transformations

$$R_{\alpha\beta} \rightarrow (Q_\alpha \otimes Q_\beta)R_{\alpha\beta}(Q_\alpha \otimes Q_\beta)^{-1} \tag{4}$$

where the Q s are non-singular ordinary matrices.

Both of the above Yang–Baxter equations (YBE) follow then from factorizability, they just differ in their labelling. And even this difference is sometimes just formal, as we will discuss next.

3. The Wu–Kadanoff duality

The vacuum (face) labelling and particle (vertex) labelling are not completely isolated; under certain restrictions (which turn out to be of practical significance) one can establish a well known duality between them [12]. Here the derivation is again done in detail because the analogies will be used later with the tetrahedron equations.

3.1. Generalities

Previously, the nature of the labels was left open, because no structure was necessary. However, now we have to do some algebra with the indices and thus we have to discuss the domain we will be working in.

Let us first recall the setting of the original lattice work. The face (vacuum) and edge (particle) labels were given by the state of a spin, either up or down. For models with more than two states the values of the state labels could be modelled by integers modulo d , or for more exotic models by pairs of integers modulo d , etc. In general we assume that the labels are elements of some Abelian group I . For spins one often uses multiplicative notation for this group, but in this paper we use additive notation.

In the following we have to do some simple arithmetic with the labels. In general, the vacuum and particle labels could be from a different Abelian group, but to simplify notation we take them to be the same, I . We will need a linear map from I to I with some parameters, we assume that the parameters belong to ring \mathcal{R} so that I will be an \mathcal{R} -module. In fact we will assume that the parameters belong to some (finite) field \mathcal{F} . The most common example is that $\mathcal{F} =$ integers modulo a prime number p , and $I = \mathcal{F}^n$.

Thus the vacuum labels of the Boltzmann weight w belong to I_v and w itself defines a map $w: I_v^{\otimes 4} \rightarrow \mathbb{C}$. Similarly the particle labels of the scattering amplitude belong to I_p and $R: I_p^{\otimes 4} \rightarrow \mathbb{C}$.

In order to define a duality relation between two labelling schemes we use the following basic principles.

(i) Hierarchy: the scattering amplitude of a labelling scheme with fewer summation indices in the factorization condition is expressed in terms of those whose equation has more summation indices.

(ii) Locality: the state of a particle is given by its nearest neighbour vacuum states.

(iii) Linearity: the above dependence is linear.

3.2. Details

The above principles imply that the duality between w and R is given by $w = R \circ f$, or explicitly (cf figure 4)

$$w(b, c, d, a) = R_{\epsilon_1 a + \tau_1 b, \epsilon_2 b + \tau_2 c}^{\epsilon_3 d + \tau_3 c, \epsilon_4 a + \tau_4 d} \tag{5}$$

with eight constants ϵ_i, τ_i , that are free at the moment (in the simplest case we expect these constants to be ± 1). This defines the local maps $f_i: I_v^{\otimes 2} \rightarrow I_p$, for example $j_1 := f_1(a, b) = \epsilon_1 a + \tau_1 b$, and the function $f: I_v^{\otimes 4} \rightarrow I_p^{\otimes 4}$ mentioned above is their extension to the domain of w , as given in (5). We must in fact extend f to the full space in question, defining $f^*: I_v^{\otimes 7} \rightarrow I_p^{\otimes 9}$, and verify that with this mapping the two equations (1) and (2) reduce to a common equation. To do the above we proceed in four steps:

(i) *Label matching.* In equation (5) there are four different pairs of ϵ and τ coefficients, but they cannot all be free, because in (2) the k_μ indices, for example, appear in different places in R and they must of course have the same expression. For example, from the two k_1 s on the LHS we conclude that $\epsilon_3 = \epsilon_1$ and $\tau_3 = \tau_1$. The final result is that the dependence on the subscript is trivial, $\epsilon_i = \epsilon, \tau_i = \tau, \forall i$, i.e. functions f_i are all alike. The extensions to f and f^* follow then straightforwardly.

(ii) *Constraints on R and w .* In (2) there are three summation indices, in (1) only one; the range of the map f^* cannot, therefore, be the full $I_p^{\otimes 9}$. We can reduce the number of summation indices only by introducing constraints of the type ' $R = 0$ for some index

values', which means that we must look at the range of f and define $R = 0$ outside. The label restriction on R must in general have the form

$$R_{ij}^{kl} = 0 \quad \text{unless} \quad Ai + Bj + Ck + Dl = 0$$

where the constants A, B, C, D are determined by f , i.e. the form of R used in (5). The labels a, b, c, d are here free and we need a non-trivial solution of

$$A(\epsilon a + \tau b) + B(\epsilon b + \tau c) + C(\epsilon d + \tau c) + D(\epsilon a + \tau d) = 0$$

on the whole $I_{\mathbb{V}}^{\otimes 4}$. This is easily found, we get $A = -D = \epsilon, B = -C = -\tau$, hence the restriction must be

$$R_{ij}^{kl} = 0 \quad \text{unless} \quad \epsilon i + \tau k = \epsilon l + \tau j. \tag{6}$$

Since we can determine the upper right label, say, of R from the others we conclude from (5) that w depends only on three indices, a convenient choice is

$$w(b, c, d, a) = \tilde{w}(\epsilon a + \tau b, \epsilon b + \tau c, d - b) \quad \text{or} \quad R_{ij}^{k*} = \tilde{w}\left(i, j, \frac{1}{\epsilon}(k - j)\right). \tag{7}$$

Thus f is one-to-one only on the three-dimensional subspaces shown in (7).

(iii) *Label conversion:* Next we must extend f to f^* and find its kernel and range. For explicit calculations we observe that the constraint (6) is coded into (5) so all its consequences are obtained when we substitute (5) into (1) and try to convert it into (2). This is easy to do: for example, in order to get the first term on the LHS of (2) right we find that we should make the substitutions

$$a = \frac{1}{\epsilon}j_1 - \frac{\tau}{\epsilon}b \quad c = \frac{1}{\tau}j_2 - \frac{\epsilon}{\tau}b \quad g = \frac{1}{\epsilon}(k_1 - j_2) + b.$$

This introduces j_1, j_2 and k_1 but then k_2 is fixed by

$$\epsilon j_1 + \tau k_1 = \epsilon k_2 + \tau j_2. \tag{8}$$

In the second term we can convert d and e into j_3 and l_1 , respectively, but k_3 is fixed by

$$\epsilon k_1 + \tau l_1 = \epsilon k_3 + \tau j_3. \tag{9}$$

These are nothing more than (6) applied to the R -term in question, but with the third term we get (after converting f into l_2) one overall restriction on the free labels

$$\epsilon^2(j_1 - l_3) - \epsilon\tau(j_2 - l_2) + \tau^2(j_3 - l_1) = 0 \tag{10}$$

which decreases the number of equations from d^6 to d^5 . Similar results are obtained on the RHS, the restrictions on the k s are now

$$\epsilon k_3 + \tau j_3 = \epsilon j_2 + \tau k_2 \quad \epsilon l_2 + \tau k_2 = \epsilon k_1 + \tau l_1 \tag{11}$$

while the condition on the free labels is again (10). Thus the range of f^* is $I_{\mathbb{P}}^{\otimes 9}$ subject to (8)–(10) on the LHS, while on the RHS the restriction is (10, 11). A one-to-one mapping is then obtained for a six-dimensional subspace, the kernel is generated e.g. by b .

(iv) *The common equation.* Finally, we can write down the common equation to which (1) and (2) reduce under (7). Using the further redefinitions $k_1 = \epsilon k + j_2$ on the LHS, $k_3 = \tau k + j_2$ on the RHS, and $l_1 = j_3 + \epsilon l'_1, l_3 = j_1 + \tau l'_3$ we find

$$\begin{aligned} \sum_k \tilde{w}(j_1, j_2, k) \tilde{w}(j_2 + \epsilon k, j_3, l'_1) \tilde{w}(j_1 + \tau k, j_2 + \epsilon k + \tau l'_1, l'_3 - k) \\ = \sum_k \tilde{w}(j_2, j_3, k) \tilde{w}(j_1, j_2 + \tau k, l'_3) \tilde{w}(j_2 + \tau k + \epsilon l'_3, j_3 + \epsilon k, l'_1 - k). \end{aligned} \tag{12}$$

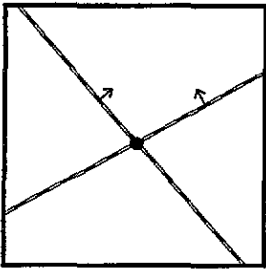


Figure 5. The basic state of two intersecting strings.

In this final equation we have resolved all conditions following from duality. The equation contains only one summation index, the functions depend only on three labels, and all the labels are free (because l_2 does not appear). Thus we have d^5 equations for d^3 unknowns. Furthermore, the reflection symmetry of (2) is preserved: the right- and left-hand sides are exchanged if we exchange the first two labels of \tilde{w} and change $1 \leftrightarrow 3$ and $\epsilon \leftrightarrow \tau$.

In the Wu-Kadanoff duality [12] one takes $\epsilon = -\tau = 1$. This means that if two adjacent cell spins point to the same direction the edge spin in between points up. For R this implies that $i + j = k + l$ is the necessary condition for a non-zero R . When the number of states is $N = 2$ and the index arithmetic is therefore (mod 2) the choice of sign does not in fact matter and we always get the eight-vertex ansatz. However, for $N > 2$ the two sign choices for $\epsilon\tau$ are genuinely different. The permutation matrix $R = \delta_i^j \delta_j^k$ is always included, but a diagonal R -matrix is included only with the above choice $\epsilon = -\tau = 1$. On the other hand, the choice $\epsilon = \tau = 1$ gives a manifestly left-right symmetric restriction.

4. The tetrahedron equation

We will now derive the tetrahedron equations for three possible labelling schemes using the above derivation of Yang-Baxter equations as a guide.

4.1. The basic state and the basic scattering process

To begin with we have a two-dimensional ambient space and in it one-dimensional objects, straight strings. But this is not yet the basic state, for it we need two intersecting strings as shown in figure 5. The two strings divide the space into four quadrants and they also cut each other in two parts, furthermore, we have a particle at the intersection of the strings. Thus to fully label the basic state we need four vacuum labels, four string labels and one particle label (along with two dynamical characterizations, the string velocity vectors, which also define the motion of the particle).

The idea that the position of a particle is defined by intersecting strings has a precedent in soliton physics. The (2 + 1)-dimensional Davey-Stewartson equation has localized solitons (called 'dromions') that are exactly of this type. It turns out that in this (and some other) soliton systems one can separate a physical and a 'ghost'-fields. The ghosts are normal plane-wave solitons but they do not show up in the physical field. However, at the intersections of the plane-wave ghosts the physical field is excited, creating an object that is localized in two dimensions [13].

For the basic scattering process we have to add one more straight string. The three strings form a triangle and the initial and final states of the basic scattering process

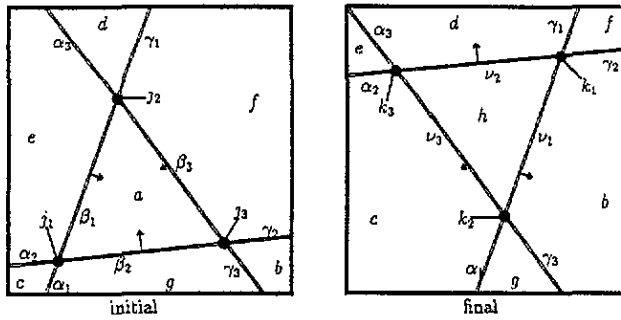


Figure 6. The initial and final states of the basic scattering process of three strings.

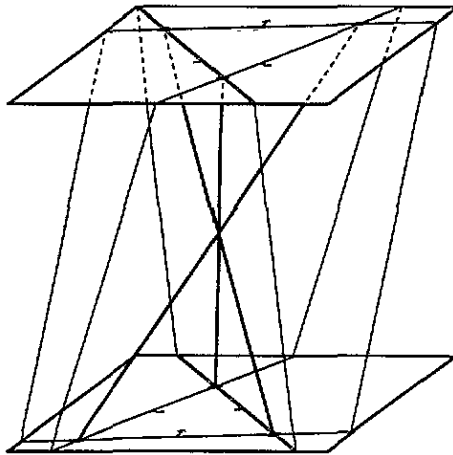


Figure 7. A perspective view of the scattering of three strings, the vertical axis is for time.

look as in figure 6. First the triangle formed by the three strings decreases to a point, turns over and then starts to expand again. A perspective view of the process with time as the vertical axis is given in figure 7. During scattering the string velocities do not change (elasticity) but the labellings of the inverting triangle can change, that is, the vacuum at the centre, the three string segments at the sides, and the three particles at the corners.

4.2. The three labelling schemes

The scattering process of figures 6 and 7 is characterized by a scattering amplitude which depends on the various labels given in figure 6. Some of the labels may of course be ignored. We will mainly discuss the following three labelling schemes:

(i) If we are only interested in the state of the vacua the scattering amplitude is given by [7]

$$w(a|efg|bcd|h) \tag{13}$$

where the rule is to write first the vacuum in the centre of the initial state (cf figure 6), then the vacua having a common edge with the centre of the initial state (clockwise, starting from the left), the vacua having a common edge with the centre of the final

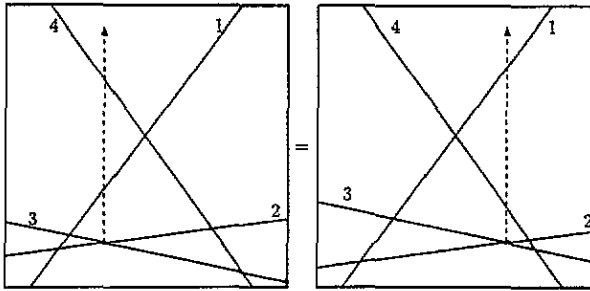


Figure 8. The two scattering scenarios with four strings. The frame of reference has been chosen so that the intersection of strings 1 and 4 is stationary. The intersection point of strings 2 and 3 can now pass the 1-4 intersection point on the left or on the right and both scenarios should give the same result.

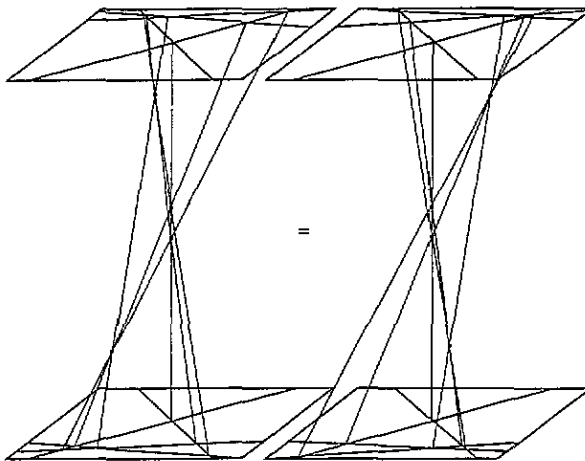


Figure 9. A three-dimensional view of the two possible scattering scenarios.

state (clockwise, starting from right) and finally the new vacuum at the centre of the final state. (Some authors use counterclockwise labelling [4].) For vacua we use lower case letters from the beginning of the alphabet. If we look at the system in three dimensions with time as the third axis then vacuum labelling can also be called volume or cell labelling.

(ii) In string (or face) labelling we write the process as

$$S \begin{matrix} \alpha_1 \beta_1 \gamma_1 \nu_1 \\ \alpha_2 \beta_2 \gamma_2 \nu_2 \\ \alpha_3 \beta_3 \gamma_3 \nu_3 \end{matrix} \quad (14)$$

where the rule is to write on each row the labels of one string, starting from the bottom of the leftmost string in the initial state and then clockwise (see figure 6). The last entry in each section is for the string segment in the middle of the new triangle in the final state. For strings we use Greek letters. (The convention in [1] is slightly different.)

(iii) In particle (or edge) labelling we write the scattering amplitude as [2]

$$R_{j_1 j_2 j_3}^{k_1 k_2 k_3} \quad (15)$$

Here the lower indices give the corners of the triangle in the initial state, (clockwise, starting from lower left-hand corner) and the upper indices refer to the triangle in the final state.

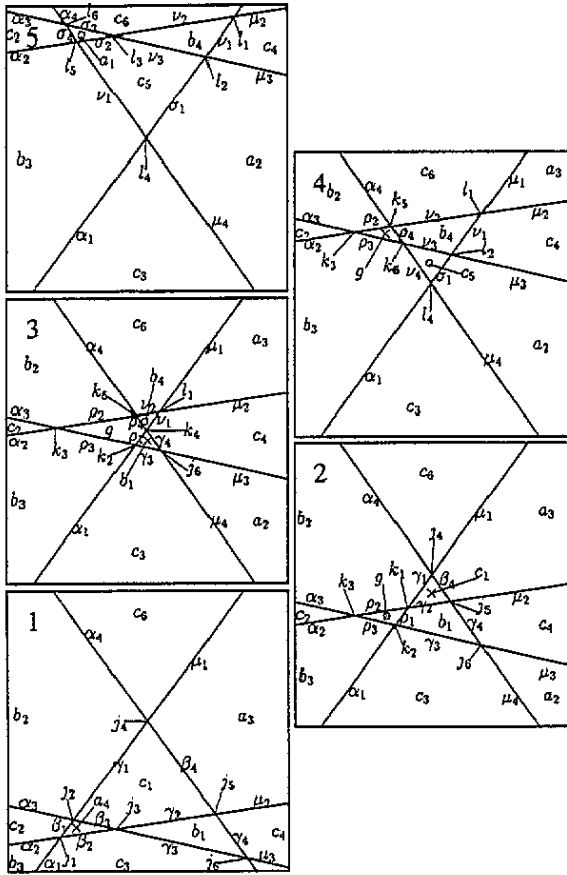


Figure 10. The sequence of five different time-slices of four string scattering corresponding to the LHS of figure 9. The triangle that is about to turn over is marked with a cross, the result with a circle. All labels within and bordering the triangle change.

For particles we use lower case letters from the middle of the alphabet.

In addition to the quantum numbers that change during interaction the scattering amplitude could also depend on non-changing ‘spectator’ quantum numbers attached to a particle or globally to a string.

4.3. The factorization condition

In analogue to section 2.2 we get a factorization condition when we add one more element into the picture. With the fourth straight string the initial state can always be rotated into the one given in figure 8; the four strings make a arrowhead-like figure that points upwards. We can now go to a frame where the top of the arrow stays fixed and observe the motion of the intersection of the other two strings. Depending on the relative initial positions of the strings this intersection can pass the stationary intersection on the left or on the right. Triple intersections take place whenever either of these intersections crosses a third string. But on the right- and left-hand sides of figure 8 the order of the triple intersections is different; the tetrahedron equation is obtained when we require that both scenarios should give the same result.

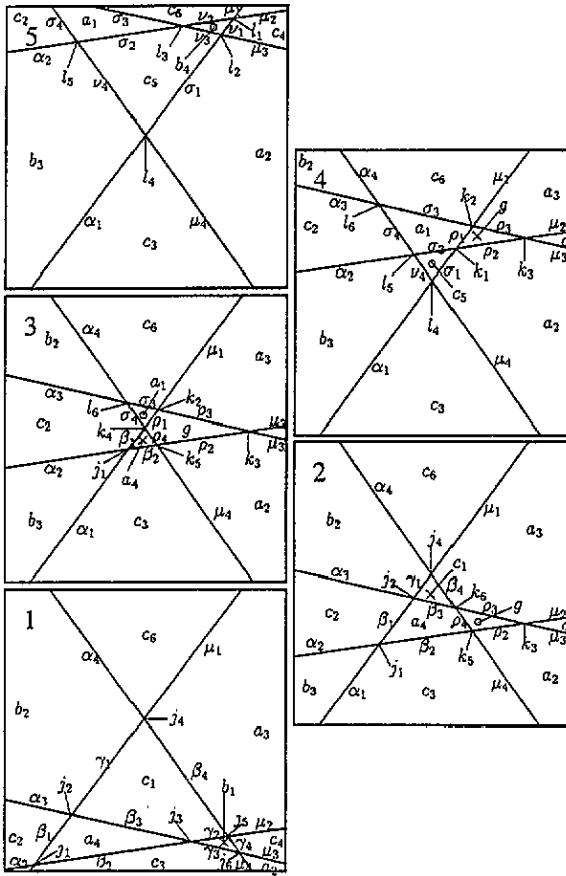


Figure 11. The sequence of time-slices corresponding to the RHS of figure 9.

A three-dimensional spacetime description of this is given in figure 9. At the centre is the tetrahedron of the tetrahedron equation, its corners have been marked by black dots. The tetrahedron has different orientation on the different sides of the equation, in the same way that the triangle is turned over in figure 4. In the following we are interested in the labelling associated with the scattering process. For that purpose it is easier to look at a sequence of two-dimensional sections rather than the three-dimensional tetrahedron, the five essentially different sections are given in figures 10 and 11.

Let us first consider the vacuum or cell labelling. Then following the diagrams of figures 10 and 11 using the convention of (13) we get the d^{14} equations (equation (2.2) of [7])

$$\begin{aligned}
 & \sum_g w(a_4|c_2c_1c_3|b_1b_3b_2|g)w(c_1|b_2a_3b_1|c_4gc_6|b_4)w(b_1|gc_4c_3|a_2b_3b_4|c_5) \\
 & \quad \times w(g|b_2b_4b_3|c_5c_2c_6|a_1) \\
 & = \sum_g w(b_1|c_1c_4c_3|a_2a_4a_3|g) \\
 & \quad \times w(c_1|b_2a_3a_4|gc_2c_6|a_1)w(a_4|c_2gc_3|a_2b_3a_1|c_5)w(g|a_1a_3a_2|c_4c_5c_6|b_4). \quad (16)
 \end{aligned}$$

If instead we use string labelling (14) we get d^{24} equations

$$\sum_{\rho_1 \rho_2 \rho_3 \rho_4} S \begin{matrix} \alpha_1 \beta_1 \gamma_1 \rho_1 \\ \alpha_2 \beta_2 \gamma_2 \rho_2 \\ \alpha_3 \beta_3 \gamma_3 \rho_3 \end{matrix} S \begin{matrix} \rho_1 \gamma_1 \mu_1 \nu_1 \\ \rho_2 \gamma_2 \mu_2 \nu_2 \\ \alpha_4 \beta_4 \gamma_4 \rho_4 \end{matrix} S \begin{matrix} \alpha_1 \rho_1 \nu_1 \sigma_1 \\ \rho_3 \gamma_3 \mu_3 \nu_3 \\ \rho_4 \gamma_4 \mu_4 \nu_4 \end{matrix} S \begin{matrix} \alpha_2 \rho_2 \nu_2 \sigma_2 \\ \alpha_3 \rho_3 \nu_3 \sigma_3 \\ \alpha_4 \rho_4 \nu_4 \sigma_4 \end{matrix} \\ = \sum_{\rho_1 \rho_2 \rho_3 \rho_4} S \begin{matrix} \beta_2 \gamma_2 \mu_2 \rho_2 \\ \beta_3 \gamma_3 \mu_3 \rho_3 \\ \beta_4 \gamma_4 \mu_4 \rho_4 \end{matrix} S \begin{matrix} \beta_1 \gamma_1 \mu_1 \rho_1 \\ \alpha_3 \beta_3 \rho_3 \sigma_3 \\ \alpha_4 \beta_4 \rho_4 \sigma_4 \end{matrix} S \begin{matrix} \alpha_1 \beta_1 \rho_1 \sigma_1 \\ \alpha_2 \beta_2 \rho_2 \sigma_2 \\ \sigma_4 \rho_4 \mu_4 \nu_4 \end{matrix} S \begin{matrix} \sigma_1 \rho_1 \mu_1 \nu_1 \\ \sigma_2 \rho_2 \mu_2 \nu_2 \\ \sigma_3 \rho_3 \mu_3 \nu_3 \end{matrix} \quad (17)$$

This is the original tetrahedron equation, (3.9) of [1], up to some label rearrangements.

Finally in the particle or edge labelling we get d^{12} equations [2]

$$R_{j_1 j_2 j_3}^{k_1 k_2 k_3} R_{k_1 j_4 j_5}^{l_1 k_4 k_5} R_{k_2 k_4 j_6}^{l_2 l_4 k_6} R_{k_3 k_5 k_6}^{l_3 l_5 l_6} = R_{j_3 j_5 j_6}^{k_3 k_5 k_6} R_{j_2 j_4 k_6}^{k_2 k_4 l_6} R_{j_1 k_4 k_5}^{k_1 l_4 l_5} R_{k_1 k_2 k_3}^{l_1 l_2 l_3} \quad (18)$$

where we have used the Einstein summation convention over the repeated k indices. We may again consider R s as matrices, which now operate non-trivially on *three* vector spaces, and use, as before, the shorthand notation of writing only the vector spaces

$$R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123} \quad (19)$$

Since each particle is at the intersections of two strings, we could as well use the pair of string labels for each particle according to the following translation table:

$$1 = (12) \quad 2 = (13) \quad 3 = (23) \quad 4 = (14) \quad 5 = (24) \quad 6 = (34) \quad (20)$$

which yields

$$R_{(12)(13)(23)} R_{(12)(14)(24)} R_{(13)(14)(34)} R_{(23)(24)(34)} \\ = R_{(23)(24)(34)} R_{(13)(14)(34)} R_{(12)(14)(24)} R_{(12)(13)(23)} \quad (21)$$

At this point we would like to discuss briefly how the Frenkel–Moore equation [3] is related to the tetrahedron labellings. This set of d^8 equations is given by

$$F_{j_1 j_2 j_3}^{k_1 k_2 k_3} F_{k_1 k_2 j_4}^{l_1 l_2 k_4} F_{l_1 k_3 k_4}^{n_1 n_2 n_4} F_{l_2 l_3 l_4}^{n_2 n_3 n_4} = F_{j_2 j_3 j_4}^{k_2 k_3 k_4} F_{j_1 k_3 k_4}^{k_1 l_3 l_4} F_{k_1 k_2 l_4}^{l_1 l_2 n_4} F_{l_1 l_2 l_3}^{n_1 n_2 n_3} \quad (22)$$

where summation is over the repeated k and l indices, or in shorthand notation

$$F_{123} F_{124} F_{134} F_{234} = F_{234} F_{134} F_{124} F_{123} \quad (23)$$

Note that when the rows in the string labelling (17) are turned into columns the index numbering in (17) and (22) is identical. In fact, it turns out that the Frenkel–Moore equation is obtained with string labelling, but the labelling is *non-local*: Let us assign to each string a global index, which changes whenever the string takes part in a triple collision. When one now goes through the sequence of scatterings in figures 10 and 11 with this convention one obtains exactly (22). Such a global scheme would not work for the Yang–Baxter equation, or for cell or particle labelling for tetrahedron equation, but for still higher-dimensional simplex equations there can be several different non-local labelling schemes. This is based on the observation that equations (21) and (23) are just two ways of distributing the indices 1234; this idea has been generalized to arbitrary dimension by Carter and Saito [5].

4.4. The spectral parameters

Above we have discussed the labels describing the quantum states of the particles, strings, or vacua. In addition there are variables related to the dynamics of the strings (often called spectral parameters) which we briefly discuss here.

The motion of a uniform string on a plane is given by a 2-vector perpendicular to the string [1]. (In principle, we should also give the initial positions of all strings at some initial time, but the basic principle in the derivation of Yang–Baxter and tetrahedron equations is that the initial position does not matter, and therefore it cannot appear as

a variable in the scattering amplitude.) In the three-dimensional picture, where moving strings are represented by planes (see figure 9) we can give the orientation of the planes by unit vectors u_i perpendicular to the plane. Thus the scattering amplitude of the basic process depends on three unit vectors and in the tetrahedron equation we have altogether four unit vectors.

The unit vectors can be represented by points on the unit sphere. Normally we are only interested in the relative positions of these points and then for the four points we need five real numbers. (A simple coordinate dependent parametrization is obtained by choosing one point as the north pole, putting the second on the 0 meridian (one coordinate), and giving the latitude and longitude for the remaining two.) A coordinate independent characterization of the relative positions can be given by the angles θ between the unit vectors, for this purpose we write

$$c_{ij} := u_i \cdot u_j = \cos(\pi - \theta_{ij}). \tag{24}$$

There are six such angles but since only five of them can be independent, as shown above, there must be constraints between them. In fact from spherical trigonometry it follows that [1]

$$\det \begin{pmatrix} 1 & c_{12} & c_{13} & c_{14} \\ c_{12} & 1 & c_{23} & c_{24} \\ c_{13} & c_{23} & 1 & c_{34} \\ c_{14} & c_{24} & c_{34} & 1 \end{pmatrix} = 0. \tag{25}$$

It should be noted that this result extends to other dimensions [4] (in the Yang–Baxter case we have a 3×3 matrix, but then the condition between the three angles u, w, v can be linearized resulting with $w = u + v$, as was used in (2)). If we now write the angle dependence explicitly (19) becomes (cf equation (21))

$$R_{123}(\theta_{12}, \theta_{13}, \theta_{23}) R_{145}(\theta_{12}, \theta_{14}, \theta_{24}) R_{246}(\theta_{13}, \theta_{14}, \theta_{34}) R_{356}(\theta_{23}, \theta_{24}, \theta_{34}) \\ = R_{356}(\theta_{23}, \theta_{24}, \theta_{34}) R_{246}(\theta_{13}, \theta_{14}, \theta_{34}) R_{145}(\theta_{12}, \theta_{14}, \theta_{24}) R_{123}(\theta_{12}, \theta_{13}, \theta_{23}). \tag{26}$$

Solutions of the tetrahedron equations with other dynamical parametrizations may also be interesting, but then one needs to discuss the meaning of these parameters. In any case the dependence on dynamical parameters is quite independent of the labelling schemes and therefore here we will not discuss this topic further.

5. Dualities between the labelling schemes

We will now construct the tetrahedron analogues of the Wu–Kadanoff duality between the three labelling schemes. We follow closely the derivation used in section 3.

The general setting is that the Boltzmann weights map as $w : I_v^{\otimes 8} \rightarrow \mathbb{C}$ while (16) itself is defined on $I_v^{\otimes 15}$, the string scattering amplitudes as $S : I_s^{\otimes 12} \rightarrow \mathbb{C}$ and (17) is defined on $I_v^{\otimes 26}$, and finally for the particle scattering amplitudes we have $R : I_p^{\otimes 6} \rightarrow \mathbb{C}$ and (18) is defined on $I_p^{\otimes 18}$. We must now construct maps F that connect the pairs as follows: $w = R \circ F_{vp}$, $w = S \circ F_{vs}$, and $S = R \circ F_{sp}$.

5.1. Duality between vacuum and particle labellings

The first step is to express the scattering amplitude w of (13) in terms of the R of (15). According to the principles of hierarchy, locality and linearity each index of R is given by

a linear combination of the vacua (cells) around the corresponding particle (edge). From figure 6 we get, after checking also that labels match (step (i) in subsection 3.2)

$$w(a|efg|bcd|h) = R_{\alpha e + \beta c + \gamma g + \delta a, \alpha d + \beta e + \gamma a + \delta h}^{\alpha d + \beta h + \gamma b + \delta f, \alpha h + \beta c + \gamma g + \delta b, \alpha d + \beta e + \gamma a + \delta h} \quad (27)$$

from which F_{vp} can be read off. The constants $\alpha, \beta, \gamma, \delta$ are the parameters of the map F_{vp} and we will later find restrictions on them.

As for step (ii), constraints on R , we observe that in (18) there are six summation indices, and in (16) only one, thus we must get at least five conditions through the four R s, that is, one condition is then not enough but we need two for each R . The constraints can only be consequences of the particular form of R in (27). Depending on the choice of parameters $\alpha, \beta, \gamma, \delta$ one can obtain linear relations among the labels of R and we can then define $R = 0$ when these relations do not hold

$$R_{ijk}^{lmn} = 0 \quad \text{unless} \quad A_1 i + A_2 j + A_3 k + B_1 l + B_2 m + B_3 n = 0.$$

Using equation (27) and requiring that these linear relations be true for all values of a, \dots, h yields eight equations. These can be easily solved for A_i, B_i under certain restrictions on $\alpha, \beta, \gamma, \delta$. There are two non-trivial results, with $VP_A: \alpha\gamma = \beta\delta$ and $VP_B: \alpha = \gamma = 0, \beta + \delta = 0$, which we now discuss separately.

5.1.1. Case VP_A with all parameters non-zero. In the first case we find that if $\alpha\gamma = \beta\delta$ we can impose two relationships between the labels of R and thus we can define

$$R_{ijk}^{lmn} = 0 \quad \text{unless} \quad \alpha l + \beta i = \beta m + \alpha j \quad \text{and} \quad \gamma m + \beta j = \beta n + \gamma k. \quad (28)$$

To get the corresponding condition for w we may assume that the l and n labels of R in (28) are determined from the others, and then we get

$$w(a|efg|bcd|h) = \bar{w}_A(\alpha e + \beta c + \gamma g + \delta a, \alpha d + \beta e + \gamma a + \delta f, \alpha f + \beta a + \gamma g + \delta b, \alpha h + \beta c + \gamma g + \delta b) \quad (29)$$

or

$$R_{ijk}^{*mn} = \bar{w}_A(i, j, k, m). \quad (30)$$

To prove that a duality exists between (16) and (18) we substitute (27) into (16) and convert it to have particle labels of (18). (We also use $\alpha\gamma = \beta\delta$ to eliminate δ .) In the process one finds restrictions on the labels, five of the eight conditions relate k_i , on the LHS as

$$\begin{aligned} \alpha k_1 + \beta j_1 &= \alpha j_2 + \beta k_2 & \gamma k_2 + \beta j_2 &= \gamma j_3 + \beta k_3 & \alpha l_1 + \beta k_1 &= \alpha j_4 + \beta k_4 \\ \gamma k_4 + \beta j_4 &= \gamma j_5 + \beta k_5 & \gamma l_4 + \beta k_4 &= \gamma j_6 + \beta k_6 \end{aligned} \quad (31)$$

and on the RHS as

$$\begin{aligned} \alpha l_1 + \beta k_1 &= \alpha k_2 + \beta l_2 & \gamma l_2 + \beta k_2 &= \gamma k_3 + \beta l_3 & \gamma l_4 + \beta k_4 &= \gamma k_5 + \beta l_5 \\ \alpha k_1 + \beta j_1 &= \alpha k_4 + \beta l_4 & \gamma k_4 + \beta j_4 &= \gamma k_6 + \beta l_6 \end{aligned} \quad (32)$$

and the remaining three restrict the unsummed indices:

$$\begin{aligned} \alpha^2(j_4 - l_1) + \alpha\beta(l_2 - j_2) + \beta^2(j_1 - l_4) &= 0 \\ \alpha\beta(j_4 - l_3) + \beta\gamma(j_3 - l_4) + \alpha\gamma(l_2 - j_5) + \beta^2(l_5 - j_2) &= 0 \\ \beta^2(j_4 - l_6) + \beta\gamma(l_5 - j_5) + \gamma^2(j_6 - l_4) &= 0. \end{aligned} \quad (33)$$

This means that the range of F_{vp}^* is ten-dimensional and thus its kernel should be five-dimensional. Indeed one finds during the label conversion that it can be generated by $\{a_1, c_1, c_2, c_3, c_6\}$.

After this it is possible to write the common equation to which (16) and (18) reduce under duality, in terms of \bar{w}_A defined in (29) or (30) it reads

$$\begin{aligned} & \sum_k \bar{w}_A(j_1, j_2, j_3, k) \bar{w}_A(j_2 + \bar{\beta}(k - j_1), j_4, j_5, l'_2 + \bar{\beta}k) \bar{w}_A(k, l'_2 + \bar{\beta}k, j_6, l_4) \\ & \quad \times \bar{w}_A(j_2 + \bar{\gamma}(k - j_3), j_4 + \bar{\gamma}(l'_2 - j_5) + \bar{\beta}\bar{\gamma}k, l'_2 + \bar{\gamma}(l_4 - j_6) + \bar{\beta}\bar{\gamma}k, l'_5 + \bar{\gamma}l_4) \\ & = \sum_k \bar{w}_A(j_3, j_5, j_6, k) \bar{w}_A(j_2, j_4, j_5 + \bar{\gamma}(k - j_6), l'_5 + \bar{\gamma}k) \bar{w}_A(j_1, l'_5 + \bar{\gamma}k, k, l_4) \\ & \quad \times \bar{w}_A(l'_5 + \bar{\gamma}k + \bar{\beta}(l_4 - j_1), j_4 + \bar{\beta}(l'_5 - j_2) + \bar{\beta}\bar{\gamma}k, j_5 + \bar{\beta}(k - j_3), l'_2 + \bar{\beta}l_4) \end{aligned} \quad (34)$$

(where we have used the further definitions $\beta = \alpha\bar{\beta}$, $\gamma = \beta\bar{\gamma}$, $l'_2 = l_2 - \bar{\beta}l_4$, $l'_5 = l_5 + \bar{\gamma}l_4$, and eliminated l_1, l_3, l_6 using (33)).

In this equation we have resolved all the conditions from vacuum-particle duality. The function \bar{w}_A depends on only four labels, and in the equation there are altogether nine free labels. Thus we have d^9 equations for d^4 functions. It has the original symmetry that the LHS and RHS are exchanged if we exchange the first and third variables of \bar{w}_A and the labels as $1 \leftrightarrow 6, 2 \leftrightarrow 5$, and change the parameters $\bar{\beta} \leftrightarrow \bar{\gamma}$.

5.1.2. Case VP_A with some zero parameters. Above we assumed that all the parameters $\alpha, \beta, \gamma, \delta$ were non-zero. The form of (28) is such that β must be non-zero (if not we must also have α or γ vanishing and then there is only one constraint). We could, however, assume that $\alpha = \delta = 0$, which yields

$$R_{ijk}^{lmn} = 0 \quad \text{unless} \quad i = m \quad \gamma m + \beta j = \beta n + \gamma k \quad (35)$$

or $\gamma = \delta = 0$, which gives

$$R_{ijk}^{lmn} = 0 \quad \text{unless} \quad j = n \quad \alpha l + \beta i = \beta m + \alpha j. \quad (36)$$

For both of these $\delta = 0$, but if $\delta \neq 0$ we can eliminate β from (28) and write the conditions as

$$\delta l + \gamma i = \gamma m + \delta j \quad \delta m + \alpha j = \alpha n + \delta k.$$

Now we get two further possibilities, for $\beta = \alpha = 0$

$$R_{ijk}^{lmn} = 0 \quad \text{unless} \quad m = k \quad \delta l + \gamma i = \gamma m + \delta j \quad (37)$$

and for $\beta = \gamma = 0$

$$R_{ijk}^{lmn} = 0 \quad \text{unless} \quad l = j \quad \delta m + \alpha j = \alpha n + \delta k. \quad (38)$$

These four possibilities are really various reflections of each other, so it is sufficient to discuss only one of them here. Let us assume that $\gamma = \delta = 0$. In deriving (31)–(33) it was assumed only that β and α were non-zero so we can use these results, the constraint (36) is also such that we can solve l and n in terms of the others. Thus the final result is just (34) with $\bar{\gamma} = 0$, i.e.

$$\begin{aligned} & \sum_k \bar{w}_0(j_1, j_2, j_3, k) \bar{w}_0(j_2 + \bar{\beta}(k - j_1), j_4, j_5, l'_2 + \bar{\beta}k) \bar{w}_0(k, l'_2 + \bar{\beta}k, j_6, l_4) \\ & \quad \times \bar{w}_0(j_2, j_4, l'_2 + \bar{\beta}k, l'_5) \\ & = \sum_k \bar{w}_0(j_3, j_5, j_6, k) \bar{w}_0(j_2, j_4, j_5, l'_5) \bar{w}_0(j_1, l'_5, k, l_4) \\ & \quad \times \bar{w}_0(l'_5 + \bar{\beta}(l_4 - j_1), j_4 + \bar{\beta}(l'_5 - j_2), j_5 + \bar{\beta}(k - j_3), l'_2 + \bar{\beta}l_4). \end{aligned} \quad (39)$$

The result simplifies considerably, and the most interesting feature here is that on the RHS the second term does *not* contain the summation index.

The other three possibilities give similar results.

5.1.3. *Case VP_B*. For this solution the parameters are restricted by $\alpha = \gamma = 0$ and $\beta = -\delta$ (for simplicity we assume $\beta = 1$, $\delta = -1$), and thus

$$w(a|efg|bcd|h) = R_{c-a,e-f,a-h}^{h-f,c-b,e-h} \quad (40)$$

and we can impose the constraints

$$R'_{ijk}{}^{lmn} = 0 \quad \text{unless} \quad m = i + k \quad j = l + n. \quad (41)$$

In this case we solve for m and j in terms of the others and define \bar{w}_B by

$$w(a|efg|bcd|h) = \bar{w}_B(c - a, a - b, h - f, e - h) \quad \text{or} \quad R'_{i*k}{}^{*n} = \bar{w}_B(i, k, l, n). \quad (42)$$

As in the previous case we can next work out the constraints this implies on the indices of the tetrahedron equation, on the LHS we get

$$k_2 = j_1 + j_3 \quad k_3 + k_1 = j_2 \quad k_4 = j_5 + k_1 \quad k_5 = j_4 - l_1 \quad k_6 = j_5 + k_1 - l_2 \quad (43)$$

and on the RHS

$$k_5 = j_3 + j_6 \quad k_6 + k_3 = j_5 \quad k_2 = l_1 + l_3 \quad k_4 + k_3 = j_2 + j_5 \quad k_1 + k_3 = l_2 \quad (44)$$

while for the unsummed indices we get

$$l_4 = j_1 + j_3 + j_6 \quad l_5 + l_2 = j_2 + j_5 \quad j_4 = l_1 + l_3 + l_6. \quad (45)$$

The common equation in terms of \bar{w}_B reads ($l_2 = l'_2 + j_5$, $j_4 = j'_4 + l_1 + l_6$)

$$\begin{aligned} & \sum_k \bar{w}_B(j_1, j_3, k, j_2 - k) \bar{w}_B(k, j_5, l_1, j'_4 + l_6) \\ & \quad \times \bar{w}_B(j_1 + j_3, j_6, j_5 + l'_2, k - l'_2) \bar{w}_B(j_2 - k, k - l'_2, j'_4, l_6) \\ & = \sum_k \bar{w}_B(j_3, j_6, j_5 - k, k) \bar{w}_B(j_2, k, j'_4 + l_1, l_6) \\ & \quad \times \bar{w}_B(j_1, j_3 + j_6, l'_2 + k, j_2 - l'_2) \bar{w}_B(l'_2 + k, j_5 - k, l_1, j'_4). \end{aligned} \quad (46)$$

This has again the symmetry exchange symmetry mentioned before, now the LHS and RHS are exchanged if we exchange the first and the second, and third and fourth variables of \bar{w}_B and the labels as $1 \leftrightarrow 6$, $2 \leftrightarrow 5$, except that $l'_2 \leftrightarrow -l'_2$.

5.2. Vacuum-string duality

To get a duality between vacuum and string labellings we assume (hierarchy!) that each string label is determined by the two neighbouring vacuum labels, from figure 6 and label matching we get (cf [7], equation (2.4))

$$w(a|efg|bcd|h) = S \frac{\epsilon c + \tau g, \epsilon e + \tau a, \epsilon d + \tau f, \epsilon h + \tau b}{\epsilon e + \tau c, \epsilon a + \tau g, \epsilon f + \tau b, \epsilon d + \tau h} \frac{\epsilon d + \tau e, \epsilon f + \tau a, \epsilon b + \tau g, \epsilon h + \tau c}{\epsilon c + \tau g, \epsilon e + \tau a, \epsilon d + \tau f, \epsilon h + \tau b} \quad (47)$$

which defines F_{vs} . Since there are only two free parameters we assume that both of them are non-zero. Using the same method as before we find that (47) implies five relations between the indices of S :

$$S \begin{pmatrix} \alpha_1 \beta_1 \gamma_1 \nu_1 \\ \alpha_2 \beta_2 \gamma_2 \nu_2 \\ \alpha_3 \beta_3 \gamma_3 \nu_3 \end{pmatrix} = 0 \quad (48)$$

unless

$$\begin{aligned} \tau \alpha_1 + \epsilon \beta_1 &= \epsilon \alpha_2 + \tau \beta_2 & \tau \beta_2 + \epsilon \gamma_2 &= \epsilon \beta_3 + \tau \gamma_3 & \tau \nu_1 + \epsilon \gamma_1 &= \epsilon \nu_2 + \tau \gamma_2 \\ \tau \alpha_2 + \epsilon \nu_2 &= \epsilon \alpha_3 + \tau \nu_3 & \epsilon^2(\alpha_3 - \gamma_1) + \epsilon \tau(\gamma_2 - \alpha_2) + \tau^2(\alpha_1 - \gamma_3) &= 0. \end{aligned}$$

This reduces the number of relevant labels from 12 to 7. The conditions seem to be much more than we need to restrict the summation in (17) to that of (16), but when one observes the way the summation indices are distributed in (17) the conditions appear more reasonable.

We can use (48) to determine $\beta_2, \gamma_1, \gamma_3, \nu_1$ and ν_3 and then we find that w depends only on seven variables, we can take, for example,

$$w(a|efg|bcd|h) = W(\epsilon c + \tau g, \epsilon e + \tau c, \epsilon d + \tau e, \epsilon e + \tau a, \epsilon f + \tau a, \epsilon f + \tau b, \epsilon d + \tau h) \tag{49}$$

or

$$S_{\substack{\alpha_1 \beta_1 ** \\ \alpha_2 * \gamma_2 \nu_2 \\ \alpha_3 \beta_3 **}} = W(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_3, \gamma_2, \nu_2). \tag{50}$$

When one now substitutes (47) into (16) and converts the vacuum labels to string labels it turns out that in the resulting expression the labels $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_3, \beta_4, \gamma_2, \mu_2, \mu_3, \nu_2, \nu_3, \sigma_3$ and one of the ρ s can be taken free, all the others are then determined by the following equations:

$$\begin{aligned} -\alpha_3 \epsilon + \beta_1 \tau - \beta_3 \tau + \epsilon \gamma_1 &= 0 & \alpha_1 \tau - \alpha_2 \epsilon + \beta_1 \epsilon - \beta_2 \tau &= 0 \\ -\beta_2 \tau + \beta_3 \epsilon - \epsilon \gamma_2 + \gamma_3 \tau &= 0 & -\epsilon \mu_1 + \epsilon \nu_2 + \mu_2 \tau - \nu_1 \tau &= 0 \\ \alpha_4 \epsilon + \beta_4 \tau - \epsilon \mu_1 - \gamma_1 \tau &= 0 & \beta_4 \epsilon - \epsilon \mu_2 - \gamma_2 \tau + \gamma_4 \tau &= 0 \\ -\epsilon \nu_1 + \epsilon \nu_3 + \mu_3 \tau - \sigma_1 \tau &= 0 & \epsilon \gamma_4 - \epsilon \mu_3 - \gamma_3 \tau + \mu_4 \tau &= 0 \\ -\epsilon \nu_2 + \epsilon \sigma_3 + \nu_3 \tau - \sigma_2 \tau &= 0 & -\alpha_1 \tau + \epsilon \nu_4 - \epsilon \sigma_1 + \mu_4 \tau &= 0 \\ -\alpha_3 \tau + \alpha_4 \epsilon - \epsilon \sigma_3 + \sigma_4 \tau &= 0 & & \end{aligned} \tag{51}$$

for the free indices, while the ρ s are related on the RHS as

$$\begin{aligned} -\epsilon \rho_1 + \epsilon \sigma_2 + \rho_2 \tau - \sigma_1 \tau &= 0 \\ -\epsilon \mu_2 + \epsilon \rho_3 + \mu_3 \tau - \rho_2 \tau &= 0 \\ -\beta_3 \tau + \beta_4 \epsilon - \epsilon \rho_3 + \rho_4 \tau &= 0 \end{aligned} \tag{52}$$

and on the LHS by

$$\begin{aligned} -\epsilon \gamma_1 + \epsilon \rho_2 + \gamma_2 \tau - \rho_1 \tau &= 0 \\ \alpha_2 \tau - \alpha_3 \epsilon + \epsilon \rho_2 - \rho_3 \tau &= 0 \\ -\alpha_4 \epsilon + \epsilon \nu_2 + \rho_2 \tau - \rho_4 \tau &= 0. \end{aligned} \tag{53}$$

These conditions mean that the d^{14} vacuum equations and d^{24} string equations collapse to ‘only’ d^{13} equations. Since the range is 14-dimensional the kernel is one-dimensional, it can be generated by c_5 .

Finally we can write down the common equation in terms of the intermediate function W of (49) or (50) as follows ($\bar{\tau} = \tau/\epsilon$):

$$\begin{aligned} \sum_k W(\alpha_1, \beta_1 + \bar{\tau}(\alpha_1 - \beta_2), \alpha_3, \beta_1, \beta_3, \gamma_2, k) & \\ \times W((k - \alpha_3)/\bar{\tau} + \beta_1 - \beta_3 + \gamma_2, k, \nu_2 + \bar{\tau}(\alpha_3 - \nu_3) + \bar{\tau}^2(\nu_4 - \beta_1) & \\ + \bar{\tau}^3(\beta_2 - \alpha_1), \alpha_3 + \bar{\tau}(\beta_3 - \beta_1), \beta_4, \beta_4 + \nu_1 - \nu_3 + \bar{\tau}(\nu_4 - \beta_3) & \\ + \bar{\tau}^2(\beta_2 - \alpha_1), \nu_2) W(\alpha_1, (k - \alpha_3)/\bar{\tau} + \beta_1 + \bar{\tau}(\alpha_1 - \beta_2), & \\ k - \alpha_3 + \nu_3 + \bar{\tau}(\beta_1 - \nu_4) + \bar{\tau}^2(\alpha_1 - \beta_2), (k - \alpha_3)/\bar{\tau} + \beta_1 - \beta_3 + \gamma_2, & \\ (\nu_1 - \nu_3)/\bar{\tau} - \beta_3 + \gamma_2 + \nu_4 + \bar{\tau}(\beta_2 - \alpha_1), \mu_3, \nu_3) & \\ \times W(\beta_1 + \bar{\tau}(\alpha_1 - \beta_2), \alpha_3, \nu_2 + \bar{\tau}(\alpha_3 - \nu_3) + \bar{\tau}^2(\nu_4 - \beta_1) + \bar{\tau}^3(\beta_2 - \alpha_1), k, & \\ k - \alpha_3 + \nu_3 + \bar{\tau}(\beta_1 - \nu_4) + \bar{\tau}^2(\alpha_1 - \beta_2), \nu_3, \sigma_3) & \end{aligned}$$

$$\begin{aligned}
 &= \sum_k W(\beta_2, \beta_3, \beta_4, \gamma_2, (\nu_1 - \nu_3)/\bar{\tau} - \beta_3 + \gamma_2 + \nu_4 + \bar{\tau}(\beta_2 - \alpha_1), \mu_3, k) \\
 &\quad \times W(\beta_1, \alpha_3, \nu_2 + \bar{\tau}(\alpha_3 - \nu_3) + \bar{\tau}^2(\nu_4 - \beta_1) + \bar{\tau}^3(\beta_2 - \alpha_1), \alpha_3 + \bar{\tau}(\beta_3 - \beta_1), \\
 &\quad \beta_4, k, \sigma_3) W(\alpha_1, \beta_1 + \bar{\tau}(\alpha_1 - \beta_2), \\
 &\quad (\sigma_3 - \nu_2)/\bar{\tau} + \nu_3 + \bar{\tau}(\beta_1 - \nu_4) + \bar{\tau}^2(\alpha_1 - \beta_2), \beta_1, (k - \beta_4)/\bar{\tau} \\
 &\quad + \beta_3, (k - \beta_4 - \nu_1 + \nu_3)/\bar{\tau} + \beta_3 + \mu_3 - \nu_4 + \bar{\tau}(\alpha_1 - \beta_2), (\sigma_3 - \nu_2)/\bar{\tau} + \nu_3) \\
 &\quad \times W((\nu_3 - \nu_1)/\bar{\tau} + \mu_3, (\sigma_3 - \nu_2)/\bar{\tau} + \nu_3, \sigma_3, \\
 &\quad (\sigma_3 - \nu_2)/\bar{\tau} - \beta_4 + k + \nu_3 + \bar{\tau}(\beta_3 - \nu_4) + \bar{\tau}^2(\alpha_1 - \beta_2), k, \\
 &\quad \beta_4 + \nu_1 - \nu_3 + \bar{\tau}(\nu_4 - \beta_3) + \bar{\tau}^2(\beta_2 - \alpha_1), \nu_2). \tag{54}
 \end{aligned}$$

5.3. String-particle duality

We proceed as before: first, by hierarchy, locality, linearity we find that S and R are related by

$$S \begin{matrix} \alpha_1 \beta_1 \gamma_1 \nu_1 \\ \alpha_2 \beta_2 \gamma_2 \nu_2 \\ \alpha_3 \beta_3 \gamma_3 \nu_3 \end{matrix} = R \begin{matrix} \xi \gamma_1 + \zeta \nu_2 + \omega \nu_1 + \eta \gamma_2, \xi \nu_1 + \zeta \nu_3 + \omega \alpha_1 + \eta \gamma_3, \xi \nu_2 + \zeta \alpha_3 + \omega \alpha_2 + \eta \nu_3 \\ \xi \beta_1 + \zeta \alpha_2 + \omega \alpha_1 + \eta \beta_2, \xi \gamma_1 + \zeta \alpha_3 + \omega \beta_1 + \eta \beta_3, \xi \gamma_2 + \zeta \beta_3 + \omega \beta_2 + \eta \gamma_3 \end{matrix} \tag{55}$$

where the 24 free constants have been reduced to 4 by label matching. This defines $F_{sp} : I_s^{12} \rightarrow I_p^6$.

Next we must find the label restriction that (55) allows; the results can be grouped into two cases, which bear interesting relation with the two cases of vacuum-particle duality.

5.3.1. Case SP_A . The first solution is obtained if $\eta = \zeta = 0$, then we can impose the condition

$$R_{ijk}^{lmn} = 0 \quad \text{unless} \quad \xi l + \omega i = \omega m + \xi j \tag{56}$$

another solution follows if we take instead $\omega = \xi = 0$, and then we get

$$R_{ijk}^{lmn} = 0 \quad \text{unless} \quad \eta m + \zeta j = \zeta n + \eta k. \tag{57}$$

One immediately recognizes these as the two parts of the case VP_A in (28): in the first case one identifies $\xi = \alpha$, $\omega = \beta$ and in the second case $\zeta = \beta$, $\eta = \gamma$. Thus the string-particle duality follows (with proper parameter choices) from vacuum-particle duality, because for the first one we need to impose only ‘half’ of the conditions of the second one.

One can now work out the conditions of either half as has been done before, but we will not do so here, because their combination as given in VP_A is in practice probably the only important one.

5.3.2. Case SP_B . The other case is obtained if we have $\zeta = -\xi$, $\eta = -\omega$, then we have the restriction

$$R_{ijk}^{lmn} = 0 \quad \text{unless} \quad \omega(m - i - k) = -\xi(j - l - n). \tag{58}$$

This is a combination of the two conditions of case VP_B (41), thus again the existence of B-type vacuum-particle duality implies the existence of a corresponding string particle duality. The detailed computations proceed as follows.

If we use (58) to determine j in terms of the other labels, we find the corresponding condition for S as

$$S \begin{matrix} \alpha_1 \beta_1 \gamma_1 \nu_1 \\ \alpha_2 \beta_2 \gamma_2 \nu_2 \\ \alpha_3 \beta_3 \gamma_3 \nu_3 \end{matrix} = \tilde{S} (\xi \beta_1 - \xi \alpha_2 + \omega \alpha_1 - \omega \beta_2, \xi \gamma_2 - \xi \beta_3 + \omega \beta_2 - \omega \gamma_3, \xi \gamma_1 - \xi \nu_2 + \omega \nu_1 - \omega \gamma_2, \\ \xi \nu_1 - \xi \nu_3 + \omega \alpha_1 - \omega \gamma_3, \xi \nu_2 - \xi \alpha_3 + \omega \alpha_2 - \omega \nu_3) \tag{59}$$

or

$$R_{i * k}^{lmn} = \tilde{S}(i, k, l, m, n). \tag{60}$$

Next we substitute (55) into (17) and convert to particle labelling. In the process we find that on the LHS the summation indices are restricted by

$$\begin{aligned} \xi(j_2 - k_1 - k_3) + \omega(-j_1 - j_3 + k_2) &= 0 \\ \xi(j_4 - k_5 - l_1) + \omega(-j_5 - k_1 + k_4) &= 0 \\ \xi(k_4 - k_6 - l_2) + \omega(-j_6 - k_2 + l_4) &= 0 \end{aligned} \tag{61}$$

and on the RHS by

$$\begin{aligned} \xi(j_5 - k_3 - k_6) + \omega(-j_3 - j_6 + k_5) &= 0 \\ \xi(-j_4 + k_2 + l_6) + \omega(j_2 - k_4 + k_6) &= 0 \\ \xi(-k_1 + k_4 - l_5) + \omega(-j_1 - k_5 + l_4) &= 0 \end{aligned} \tag{62}$$

and on the free labels on both sides we get one condition

$$\xi^2(j_4 - l_1 - l_3 - l_6) + \xi\omega(-j_2 - j_5 + l_2 + l_5) + \omega^2(j_1 + j_3 + j_6 - l_4) = 0. \tag{63}$$

This means that the range of F_{sp}^* is 14-dimensional, and hence the kernel 10-dimensional. Indeed we find that the kernel is generated by $\{\alpha_4, \beta_3, \beta_4, \gamma_4, \rho_4, \nu_3, \nu_4, \mu_2, \mu_3, \mu_4\}$.

Finally we can write the common equation in terms of \tilde{S} using (63) to eliminate j_4 :

$$\begin{aligned} \sum_{k_1, k_3, k_5} \tilde{S}(j_1, j_3, k_1, j_1 + j_3 + \frac{\xi}{\omega}(k_1 + k_3 - j_2), k_3) \tilde{S}(k_1, j_5, l_1, k_1 + k_3 + k_6 + l_2 \\ - j_2 + \frac{\omega}{\xi}(j_1 + j_3 + j_6 - l_4), l_3 + l_6 + \frac{\omega}{\xi}(k_3 + k_6 - l_5)) \\ \times \tilde{S}(j_1 + j_3 + \frac{\xi}{\omega}(k_1 + k_3 - j_2), j_6, l_2, l_4, k_6) \tilde{S}(k_3, k_6, l_3, l_5, l_6) \\ = \sum_{k_2, k_3, k_6} \tilde{S}(j_3, j_6, k_3, j_6 + j_3 + \frac{\xi}{\omega}(k_6 + k_3 - j_5), k_6) \\ \times \tilde{S}(j_2, k_6, l_3 + l_1 + \frac{\omega}{\xi}(k_3 + k_1 - l_2), k_1 + k_3 + k_6 + l_5 \\ - j_5 + \frac{\omega}{\xi}(j_1 + j_3 + j_6 - l_4), l_6) \\ \times \tilde{S}(j_1, j_6 + j_3 + \frac{\xi}{\omega}(k_6 + k_3 - j_5), k_1, l_4, l_5) \tilde{S}(k_1, k_3, l_1, l_2, l_3). \end{aligned} \tag{64}$$

This again has the reflections symmetry of the original equation.

6. Simultaneous dualities between all pairs

Above we have analysed the dualities between pairs of labelling schemes. In this section we will discuss the possibility of a having simultaneously dualities between two or even three pairs of labellings. It is natural to assume that such pairs of dualities would imply relations between the parameters used to define the maps F , and we will try to satisfy any further conditions primarily by imposing relation on these parameters.

Let us first consider simultaneous existence of vacuum-particle and vacuum-string duality. For both dualities the conditions for their existence can be written as a condition on how the Boltzmann weights actually depend on the vacuum labels, (29) or (42) versus

(49). From these we can see that it will be possible to satisfy the requirements for vacuum-string duality as a consequence of vacuum-particle duality, if the right-hand side of (29) (or equation (42)) can be expressed in terms of the combinations $\epsilon\tau + \tau g$ etc on the right-hand side of (49). For case VP_A (for which $\alpha\gamma = \beta\delta$) it turns out that this can indeed be done, provided that

$$(\alpha\tau - \beta\epsilon)(\beta\tau - \gamma\epsilon) = 0. \quad (65)$$

This is a compatibility condition on the parameters used in the definition of the dualities and as said before we will first implement them. If equation (65) holds and we have vacuum-particle duality, we can then impose the restriction $\bar{W} = \bar{w}_A$ and get at the same time vacuum-string duality. For example, if we use the first factor of (65) we get

$$\begin{aligned} W(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = & \bar{w}_A((\beta\tau x_2 + \epsilon\gamma(x_4 - x_2) + \gamma\tau x_1)/\tau^2, (\beta x_3 + \gamma x_5)/\tau, \\ & (\beta\tau x_5 + \epsilon\gamma(x_4 - x_2 - x_5 + x_6) + \gamma\tau x_1)/\tau^2, \\ & (\beta\epsilon(x_7 - x_3) + \beta\tau x_2 + \epsilon\gamma(x_4 - x_2 - x_5 + x_6) + \gamma\tau x_1)/\tau^2) \end{aligned} \quad (66)$$

and this also implies that (54) becomes (34) (after a change of labels). For case VP_B it turns out that no extra conditions on parameters are needed: \bar{w}_B of (42) is automatically a function of the variables in W and we can impose

$$\begin{aligned} W(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = & \bar{w}_B((x_2 - x_4)\beta/\tau, (x_5 - x_6)\beta/\tau, \\ & (\epsilon(x_7 - x_3) + \tau(x_4 - x_5))\beta/(\epsilon\tau), (x_3 - x_7)\beta/\tau) \end{aligned} \quad (67)$$

and as a consequence (54) reduces to (46). Thus in either case a vacuum-string duality can be obtained, once the vacuum-particle duality has been established.

For vacuum-particle and string-particle dualities the condition can be best written on R , (28) or (41) versus (56), (57) or (58). But as we have already observed in section 5.3, the first duality implies the second one when the parameters are chosen properly.

Next let us consider simultaneous vacuum-string and string-particle dualities. Here we are interested in the composition $F'_{vp} = F_{sp} \circ F_{vs}$, which defines a map from vacuum to particle labels. It can always be made and it is then interesting to see how it is related to the already discussed dualities of VP_A and VP_B . In the first case of SP_{A1} we put $\eta = \zeta = 0$ and then it turns out that the resulting F'_{vp} is the same as that of VP_A if we identify

$$\eta = \zeta = 0: \quad \alpha = \epsilon\xi \quad \beta = \epsilon\omega \quad \gamma = \omega\tau \quad \delta = \tau\xi. \quad (68)$$

In particular, we recover $\alpha\gamma = \beta\delta$, which is the defining relation of the map VP_A . For the other case SP_{A2} we have to put $\omega = \xi = 0$ and then VP_A again follows when we identify

$$\omega = \xi = 0: \quad \alpha = \epsilon\zeta \quad \beta = \tau\zeta \quad \gamma = \eta\tau \quad \delta = \epsilon\eta. \quad (69)$$

Recall that each of the two choices of SP_A yielded only one of the two condition of VP_A , the extra restriction from simultaneous vacuum-string duality has now somehow created the missing one. Recall also that the simultaneous existence of vacuum-particle and vacuum-string dualities is possible for case VP_A if condition (65) holds, it now turns out that the first factor of (65) vanishes automatically for (69), the second for (68). Finally, when the string-particle duality is given by SP_B we first put $\eta = -\omega$ and $\zeta = -\xi$ and then the composed map is that of VP_B (where we have $\alpha = \gamma = 0$), if

$$\eta + \omega = \zeta + \xi = \alpha = \gamma = 0: \quad \beta = (\epsilon\omega - \tau\xi) \quad \delta = -(\epsilon\omega - \tau\xi). \quad (70)$$

Note, in particular, that in this case we get a non-vanishing condition on the parameters, we must assume that $\epsilon\omega - \tau\xi \neq 0$.

In summary, if the conditions of A- or B-type vacuum-particle duality are satisfied one can always find compatible parameter values for the vacuum-string and string-particle dualities so all three can coexist.

7. Examples

The original solution of Zamolodchikov [1] was for a two-state model given in the string labelling. It passes conditions for A-type duality, and indeed, the detailed studies of Baxter [7] were done in the vacuum labelling, and in [4] this solution was written in the particle labelling. This solution has full spectral parameter dependence, with the determinantal relation (25) and therefore checking the solution is quite tedious, see [7] for details.

In [9] Korepanov gave a two state solution that is very close to passing the requirements (41) of B-type duality: instead of (41) R is non-zero when $m = i + k + (i + l)(k + n)$ and $j = l + n + (i + l)(k + n)$. We present here a slight modification of Korepanov's ansatz which does pass the duality conditions; in this new solution the non-zero components of $R_{(ij)(ik)(jk)}$ are as follows:

$$\begin{aligned}
 R_{000}^{000} = R_{011}^{011} = R_{101}^{101} = R_{110}^{110} = 1 & \quad R_{111}^{001} = R_{001}^{111} = R_{010}^{100} = R_{100}^{010} = \frac{s_{ij}}{s_{ik}} \\
 R_{001}^{010} = R_{010}^{110} = -R_{111}^{100} = -R_{100}^{111} = \frac{s_{jk}}{s_{ik}} & \quad R_{101}^{000} = R_{011}^{011} = -R_{011}^{110} = -R_{000}^{101} = \epsilon s_{ij} s_{jk}
 \end{aligned} \tag{71}$$

where $\epsilon = \pm$ and the s_{ij} s just have to satisfy

$$s_{ij}^2 s_{ik}^2 s_{jk}^2 = s_{ij}^2 - s_{ik}^2 + s_{jk}^2 \tag{72}$$

which is solved by Korepanov's parametrization

$$s_{ij}^2 = \tanh(\phi_i - \phi_j). \tag{73}$$

The 'spectral parameter' ϕ_i is attached to string i and could, for example, be related to the string velocity, rather than to the spherical angles as was the case with Zamolodchikov's solution.

8. Discussion

In the first part of this paper we have derived, from the string scattering point of view, three kinds of tetrahedron equations corresponding vacuum (cell) labelling (16), string (face) labelling (17) and particle (edge) labelling (18).

In the second part we have studied the condition under which analogues of Wu-Kadanoff duality can be established between these labelling schemes. The final result is that there are two essentially different ways this can be done, and then with proper choice of the map parameters it is in fact possible to have a simultaneous duality between each of the three pairs of labellings. In terms of R the conditions and resulting equation can be obtained already from vacuum-particle duality and for the two alternatives they are given in (28) and (41). The other dualities then follow if we choose the remaining parameters properly, i.e. according to (69)-(70). This is a very satisfactory result: the labelling schemes and dualities form a coherent structure where everything fits together well.

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